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Technical Notes

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Induced Drag of Wings of Finite Aspect Ratio

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Introduction

THE first qualitative description of the flow passing a finite aspect ratio wing and the appearance of a trailing vortex sheet was published by Lanchester.[1] Formation of a trailing vortex sheet behind the wing is inevitable due to the pressure differences in the spanwise direction, as a result of generating lift, around the wing. Such a vortex sheet is in general unstable and readily rolls up into a pair of vortices with opposite sense of rotation at some distance downstream.[2] To render the complicated three-dimensional wing flow solvable, Prandtl[3] introduced the idea of spanwise loading and a flat continuous sheet extending far downstream to infinity for the deformed trailing vortex. Consequently, use can be made of the two-dimensional aerofoil theory and the two-dimensional boundary-layer theory. The existence of the trailing vortex sheet induces a local downwash w_i along the wing which is a strong function of strength of the trailing vorticity and its distribution in space. However, many experimental observations reveal that the behavior of a trailing vortex sheet behind a lifting wing undergoes two basic processes, as explained thoroughly by Batchelor[4]: 1) the rolling-up of the sheet, and 2) the lateral deformation of the region of nonzero vorticity in a plane normal to the direction of the freestream. These two processes, as a meaningful comparison with experiments, must be further sought in their dependence on the downstream distance. The rolling-up process is inevitable under the action of the induced velocity of the vortex sheet and usually involves a downstream length scale of an order of four wing spans for its completion. The second seems to occur at a downstream distance of average two or three mean chords with considerable concentration of vorticity in most test cases.

One of the main effects of the concentration of vorticity exhibited by a trailing vortex sheet with downstream distance is the tendency of the redistribution of the vorticity in space and the consequence of the necessary variation in the velocity field brought about by such a deformed vortex system. The contribution of the redistribution on the magnitude of the induced drag remains an interesting question to answer and is of practical importance, as noted earlier by Lam.[5]

Model for Lifting Wakes

We define an Eulerian right-handed coordinate system (x, y, z): the ox axis is taken parallel to the freestream direction U, the oy axis is to starboard, and the oz axis is upward. Let $Z(x, y)$ be the wing shape situated in the $z = 0$ plane. The perturbation velocity potential Φ satisfies the Laplace equation

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0 \tag{1}$$

with the boundary conditions on the wing Ξ and the wake Σ:

$$\begin{gathered}\Phi = 0, \quad x = -\infty, \quad y = \pm\infty, \quad z = \pm\infty \\ w(x, y, z) = (U + u)\frac{\partial Z}{\partial x} + v\frac{\partial Z}{\partial y} \text{ on } \Xi \\ |\nabla\Phi| < \infty \\ \frac{\partial\Phi}{\partial x} = 0 \text{ on } \Sigma\end{gathered} \tag{2}$$

We are mainly concerned with the antisymmetric solutions with wings of symmetric loading for a given wake model, which results in the information required for the evaluation of the overall aerodynamic forces. Within the framework of the linearized theory for the equations of motion, the wake vortex sheet is taken, after imposing the Kutta-Joukowski condition at the trailing edge of a wing of semispan b, to be confined in a plane Σ where, with the sheet surface generator being parallel to the direction of U,

$$\Sigma \in z = 0, \qquad -b \le y \le b, \qquad x \ge 0 \tag{3}$$

The irrotational flow regime outside the domain just defined can be trivially proven to satisfy the Kelvin's circulation theorem from the start of the motion and afterward. The wake consists of a series of piecewise continuous vorticity segments in the direction of motion. In addition to the relevance of a model to a real flow situation, the actual distribution of the trailing vorticity in the region must be conformal to the following three fundamental relations for any given dispersion of the shedding vorticity (with the exception of the part in the vicinity of the trailing edge) in any plane normal to the direction of motion. We suppose these three conditions are stationary with respect to time and the downstream distance: 1) constancy of the impulse required to generate the motion of the irrotational flow instantaneously from rest, 2) constancy of the moment of the impulse just mentioned about the axis of symmetry of the lifting body, and 3) invariance of the kinetic energy associated with such a motion generated from rest under the action of the impulse. The first condition is directly determined by the Joukowski theorem of lift and circulation around a lifting body with a uniform translation through an externally unbounded irrotational flowfield. The last requirement can be established readily through the proportional relationship between the total kinetic energy of the flowfield with the total impulse required to generate such a motion from rest. Consequently, discontinuity in the vorticity, such as abruptly bent vortex lines, is allowed; i.e., see Ref. 6. Determination of the induced velocity allows the calculation of the drag to be possible with a given circulation distribution. In general, the in-

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duced velocity at a point $P(x', y', z')$ on the wing planform S in the $z=0$ plane is given by the Biot-Savart law as

$$w(x', y', 0) = -\frac{1}{4\pi}\sum_{j=1}^{M} \mathrm{CP}\iint_S \frac{1}{y-y'}\left[\gamma_j g_j + \frac{\gamma_0\sqrt{(x-x')^2+(y-y')^2}}{x-x'}\right] \mathrm{d}x\,\mathrm{d}y \quad (4)$$

where $\gamma_0(x,y)$ denotes the wing spanwise circulation distribution and CP stands for the Cauchy principal value. $\gamma_j(x,y)$ and $g_j(x,y)$ stand for the vorticity and geometry function for each wake segment. The first part under the integration on the right-hand side, which is the main concern in the present study, gives rise to the downwash component responsible for the induced drag and the second one is the velocity induced by the bound circulation. Prandtl's theory corresponds to the condition of $\gamma_0=\gamma_1=\cdots=\gamma_M$ and $g_j=1$. The relevant velocity potential is given by

$$\Phi = \frac{1}{4\pi}\sum_{j=1}^{M}\iint_S \frac{z}{(y-y')^2+z^2}\left[\gamma_j g_j + \frac{\gamma_0(x-x')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right]\mathrm{d}x\,\mathrm{d}y \quad (5)$$

Application of the Wake Model

The most important application of the theory is to calculate the downwash by the rolling up of vortex sheets behind wings of normal loading, say $\Gamma(y)$. By considering one-half of a symmetrically loaded wing of semispan b, the mean effect, of redistribution of the sheet is modeled by setting $M=2$ with the following conditions (Fig. 1):

1) The trailing vortex sheet, whose strength $\gamma_1 = -\mathrm{d}\Gamma/\mathrm{d}y$, is assumed to have rolled up instantaneously at some distance l downstream (being nonzero).

2) m_k discrete concentrated line vortices are formed thereafter to infinity at the lateral locations defined by

$$b_k = \frac{\int_{yk}^{y_{k+1}} y\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\,\mathrm{d}y}{\int_{yk}^{y_{k+1}}\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\,\mathrm{d}y} \quad \text{with} \quad k=0,1,2\cdots m_k \quad (6)$$

where y_k lie between the wingtip y_0 and the median span y_{m_k} and satisfy the conditions of

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y}=0, \qquad \frac{\mathrm{d}^2\Gamma}{\mathrm{d}y^2}>0 \quad (7)$$

3) The strength of these line vortices is calculated according to

$$\Gamma_k = \int_{yk}^{y_{k+1}}\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\,\mathrm{d}y \quad (8)$$

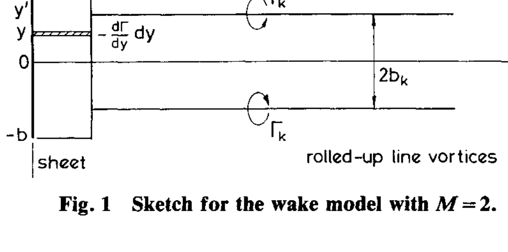

Fig. 1 Sketch for the wake model with $M=2$.

The downwash on the wing due to the sheet at y on a point y' is evaluated by choosing the geometry functions to be

$$g_1 = \frac{l}{(y-y')\sqrt{(y-y')^2+l^2}} \quad (9)$$

$$g_2 = \sum_{k=0}^{m_k}\left[\left(1-\frac{l}{\sqrt{(b_k+y')^2+l^2}}\right)+\left(1-\frac{l}{\sqrt{(b_k-y')^2+l^2}}\right)\right] \quad (10)$$

The spanwise circulation by the Glauert circulation function[7] is defined as

$$\Gamma(\theta) = 4bU_\infty\sum_{n=1}^{\infty}A_n\sin(n\theta) \quad (11)$$

and use is made of the dimensionless notations,

$$y=-b\cos\theta, \qquad y'=-b\cos\phi$$
$$\lambda = l/b, \qquad \mu_k = b_k/b \quad (12)$$

The downwash equations are simplified as

$$w_1 = \frac{U_\infty}{\pi}\sum_{n=1}^{\infty} nA_n\,\mathrm{CP}\int_0^{\pi}\frac{\lambda\cos(n\theta)\,\mathrm{d}\theta}{(\cos\theta-\cos\phi)\sqrt{\lambda^2+(\cos\theta-\cos\phi)^2}} \quad (13)$$

$$w_2 = \sum_{k=0}^{m_k}\frac{\Gamma_k U_\infty}{\pi}(\bar{w}_{k_+}+\bar{w}_{k_-}) \quad (14)$$

where

$$\bar{w}_{k_\pm} = -\int_0^{\pi}\frac{\delta(\cos\theta\pm\mu_k)}{\cos\theta\pm\cos\phi}\left(1-\frac{\lambda}{\sqrt{\lambda^2+(\cos\theta\pm\cos\phi)^2}}\right)\sin\theta\,\mathrm{d}\theta \quad (15)$$

where δ is the Dirac δ function. It has been shown that[5]

$$w_1 = \frac{U_\infty}{\pi}\sum_{n=1}^{\infty}nA_n\left[\left(\frac{2\lambda}{\sigma}E(\kappa)\right)\frac{\sin(n\phi)}{\sin\phi}+PI_n\right] \quad (16)$$

where

$$\sigma^2 = \sqrt{[(1+\cos\phi)^2+\lambda^2][(1-\cos\phi)^2+\lambda^2]}$$
$$\kappa^2 = \frac{1}{2}\left(1+\frac{1-\cos^2\phi-\lambda^2}{\sigma^2}\right) \quad (17)$$

and $E(\kappa)$ is the complete elliptic integral of the first kind with modulus κ. PI_n denotes a particular integral and is only nonzero for nonelliptic load distributions.

Results and Discussion

We apply the model to wings with elliptic loading. The induced drag is calculated from

$$D_i = -\sum_{j=1}^{2}\int_0^{\pi}\rho w_j\Gamma_j\,\mathrm{d}\theta = -\sum_{j=1}^{2}\int_0^{\pi}\mathrm{d}_{ij}\,\mathrm{d}\theta \quad (18)$$

Figure 2 shows the total downwash as the downstream distance λ varies. The rolled-up line vortices alter the distribution of the downwash velocity substantially. The strong influence of the rolled-up vortices at relatively lower values of λ is a consequence of the fact that the perturbation velocity decays with $1/r^2$, where r is the distance. Practical cases of high angle of incidence and lightly loaded with poorly designed wingtips tends to accelerate the rate of roll-up with downstream dis-

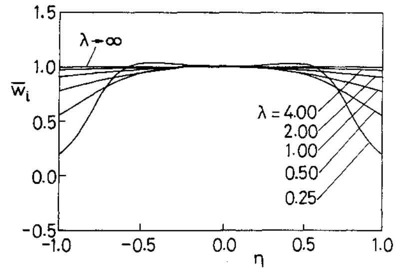

Fig. 2 Total downwash induced by the trailing vortex system over a range of values of λ for the elliptic loading $\bar{\Gamma}_1 = \sin\theta$. The downwash $\bar{w}_i$ has been normalized by $U_\infty A_1$. This value is equal to the constant downwash for the elliptic loading in Prandtl's classical theory for high aspect ratio wings.

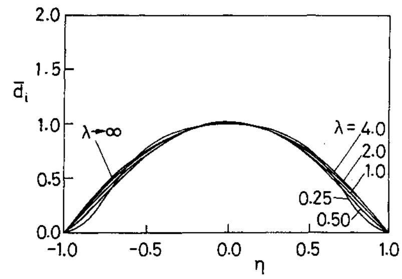

Fig. 3 Distribution of the induced drag. The drag components $\bar{d}_i$ have been normalized by $4\rho b^2 U_\infty^2 A_1^2$. This value is the least drag predicted by Prandtl's theory. The distribution of the least drag is also plotted for comparison.

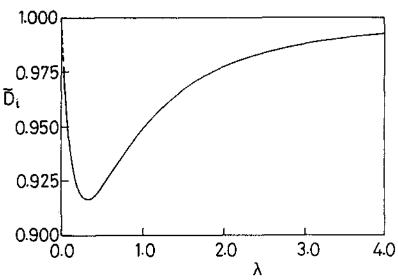

Fig. 4 Total induced drag as a function of the downstream distance λ, $\lambda_o = 0.33$.

tance. The progressive changes in the upwash toward the tips are particularly interesting for increasing values of λ. Prandtl's original results may be recovered as an asymptotic limit of the present model when the rolling-up process would have completed after a downstream distance of two wing spans. It is interesting to compare with the analytic results of Kaden[8] that the downstream distance x_r of complete roll up is given by $x_r/b \sim 0.56(A/C_L)$. This distance equals two and four spans for C_L being 1.0 and 0.5, respectively, for a wing of aspect ratio 7. The integrand of the induced drag $\bar{d}_i$ illustrates the distribution of the drag across the wing span as the parameter λ varies (Fig. 3). The variation in the drag distribution at lower values of λ is particularly noticeable, since the integrated values represent the measure of the induced drag. The vanishing drag at the wingtips is a direct result of the assumption of zero circulation at these points. The relation of the drag with the total lift is implicitly contained in the first coefficient A_1. Figure 4 shows one of the most important conclusions derived from the present study: For the loading case examined, there is an optimum downstream distance λ_o for completion of the rolling up, which results in a minimum induced drag meaningfully lower than the classical limit. Note that the unity indicates the classical value of the least drag. This potential energy picture immediately reminds us of the analogous potential of the interaction between atoms existing among the microscopic structure of matter. At short distance of λ, the downwash is dominant by the repulsive effect of the concentrated line vortices, whereas the attractive influence of the planar vortex sheet manifests itself at large distance of λ. The minimum drag attains when the effect of the concentrated line vortices is in an equilibrium state with the influence of the vortex sheet in the similar way as what happens in the achievement of the neutral position between repulsion and attraction among the atoms of matter.

Conclusions

An improved lifting-line theory for inviscid flow, taking into account some mean effect of trailing vortex sheet roll up with the downstream distance, has been developed. Calculations based on the wing wake model show that, for a given load distribution commonly encountered in practice, the roll up of the sheet within a downstream distance comparable to the wing span results in a reduced induced drag relative to the value evaluated by the classical lifting-line theory of Prandtl. This benefit comes from the variation in the downwash velocity field brought about by the rolled-up line vortices.

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Two-Equation Turbulence Model for Compressible Reacting Flows

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Introduction

ONE of the major areas of current interest where computational fluid dynamics (CFD) is used extensively is the development of advanced air-breathing propulsion systems for hypersonic vehicles. A hydrogen-fueled supersonic combus-

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